ON RESTRICTING CAUCHY-PEXIDER EQUATIONS TO SUBMANIFOLDS

MARCOS CHARALAMBIDES

Abstract. Sufficient geometric conditions are given which determine when the Cauchy-Pexider functional equation $f(x)g(y) = h(x+y)$ restricted to x, y lying on a hypersurface in $\mathbb{R}^d$ has only solutions which extend uniquely to exponential affine functions $\mathbb{R}^d \to \mathbb{C}$ (when f, g, h are assumed to be measurable and non-trivial). The Cauchy-Pexider-type functional equations $\prod_{j=0}^{d} f_j(x_j) = F(\sum_{j=0}^{d} x_j)$ for $x_0, \ldots, x_d$ lying on a curve and $f_1(x_1)f_2(x_2)f_3(x_3) = F(x_1+x_2+x_3)$ for x_1, x_2, x_3 lying on a hypersurface are also considered.

1. Introduction

The restricted Cauchy-Pexider functional equation is

$$f(x)g(y) = h(x+y) \text{ for all } (x,y) \in S \tag{1.1}$$

where S is some subset of $\mathbb{R}^d \times \mathbb{R}^d$, $d \geq 1$ and f, g and h are complex-valued functions defined on suitable domains.

The case when (1.1) holds for $S = \mathbb{R}^d \times \mathbb{R}^d$ is the classical Cauchy-Pexider equation; see [1] for an introduction to this and other related functional equations. In this case, if f, g, h are measurable functions which vanish only on sets of measure zero then there exist $v \in \mathbb{C}^d$ and $c_1, c_2 \in \mathbb{C}$ such that $f(x) = c_1 e^{v \cdot x}$ and $g(x) = c_2 e^{v \cdot x}$ for Lebesgue-almost every $x \in \mathbb{R}^d$.

For $A, B \subset \mathbb{R}^d$ and $n \in \mathbb{N}$, define the sumsets $A + B = \{a + b \mid a \in A, b \in B\}$ and $nA = \{a_1 + \ldots + a_n \mid a_1, \ldots, a_n \in A\}$. Write $v \cdot x$ for the sum $\sum v_j x_j$ whenever $v \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$. Denote the open ball of radius r and centre c in $\mathbb{R}^d$ by $B_r(c)$. If U is a set in an underlying topological space (which will be clear from context), write $\overline{U}$ for its closure. If $(X, \mathcal{M}, \mu)$ is a measure space, we will say that S is μ-null (respectively μ-full) if $S \in \mathcal{M}$ and $\mu(S) = 0$ (respectively $\mu(X \setminus S) = 0$).

Consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with induced surface measure σ. In Christ-Shao [2] it is shown that in the case when (1.1) holds for a σ^2-full set $S \subset \mathbb{S}^2 \times \mathbb{S}^2$, where $f : \mathbb{S}^2 \to \mathbb{C}$, $g : \mathbb{S}^2 \to \mathbb{C}$, $h : \overline{B_2(0)} \to \mathbb{C}$ are measurable and f and g vanish only on σ-null sets, it follows that f and g must be of the form $f(x) = c_1 e^{v \cdot x}$, $g(x) = c_2 e^{v \cdot x}$ for σ-almost every $x \in \mathbb{S}^2$.

This phenomenon can fail. Consider first the case of the parabola $P \subset \mathbb{R}^2$ consisting of all points of the form (x, x^2) for $x \in \mathbb{R}$. Then, given any non-vanishing measurable function $s : \mathbb{R} \to \mathbb{R}$, the functions f, g, h defined respectively on P, P

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and $2P$ by $f(x) = g(x) = s(x)$ and $h(x + y) = s(x)s(y)$ are measurable and satisfy the functional equation

$$f(x)g(y) = h(x + y) \text{ for all } (x, y) \in P^2.$$

Indeed, the only issue is whether h is well-defined. But it is straightforward to check that each $z \in 2P$ has a unique, up to order, decomposition as $z = x + y$ for $x, y \in P$. This example can be used to obtain higher-dimensional counterexamples. For instance, consider the hypersurface $C = P \times \mathbb{R} \subset \mathbb{R}^3$. Given any non-vanishing measurable function $s : \mathbb{R} \to \mathbb{R}$, the functions f, g, h defined respectively on C, C and $2C$ by $f(\mathbf{x}) = g(\mathbf{x}) = s(x_2)$ and $h(\mathbf{x}+\mathbf{y}) = f(\mathbf{x})g(\mathbf{y}) = s(x_2)s(y_2)$ are measurable and satisfy the functional equation (1.1) with $S = C \times C$.

In the sequel, submanifolds of $\mathbb{R}^d$ will always be smooth embedded submanfolds of $\mathbb{R}^d$ (but we do not assume that they are compact or closed) and we will refer to a codimension 1 submanifold of $\mathbb{R}^d$ as a hypersurface. A submanifold M of $\mathbb{R}^d$ comes equipped with the induced measure μ associated to the Riemannian structure on M induced by the standard Euclidean structure on $\mathbb{R}^d$. To the sumset $nM \subset \mathbb{R}^d$ we associate the n-fold convolution measure $\mu * \cdots * \mu$ which is given by

$$\mu * \cdots * \mu(E) = \mu^n(\{(x_1, \ldots, x_n) \in M^n \mid x_1 + \cdots + x_n \in E\}).$$

For $d \geq 3$, a *cylinder* is defined to be a hypersurface which is, up to rigid motions, an open subset of $\Gamma \times \mathbb{R}^{d-2}$ for some smooth embedded plane curve $\Gamma \subset \mathbb{R}^2$.

Let M be a hypersurface with induced measure μ. The main result of this paper is that, unless M is flat somewhere or contains a cylinder, whenever the non-vanishing measurable functions f, g, h satisfy (1.1) for μ^2-a.e. pair (x, y), it follows that f and g must be of the form $f(x) = c_1 e^{v\cdot x}$, $g(x) = c_2 e^{v\cdot x}$ μ-almost everywhere and, furthermore, c_1, c_2, v are uniquely determined.

If M is orientable, let $\mathcal{G} = \mathcal{G}^M : M \to \mathbb{S}^{d-1}$ denote the smooth Gauss normal map, where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. Observe that for any hypersurface M, whether the linear map $(d\mathcal{G}^U)_x$ is zero for a point $x \in M$ and some orientable open neighbourhood $U \subset M$ of x is independent of the choice of U.

Definition 1.1. A hypersurface M is *nowhere-flat* if for every $x \in M$ there exists an orientable open neighbourhood U of x such that $(d\mathcal{G}^U)_x$ is non-zero. M is *cylinder-free* if it does not contain a cylinder.

Note that M is nowhere-flat if and only if at each point $x \in M$ at least one of the principal curvatures is non-zero. Hypersurfaces in $\mathbb{R}^d$ for $d \geq 3$ which are nowhere-flat and cylinder-free include the cone $\{(x, t) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid t = \|x\|, t > 0\}$ and any hypersurface of non-vanishing Gaussian curvature.

Theorem 1.1. *Let $d \geq 3$. Let $M \subset \mathbb{R}^d$ be a connected hypersurface with induced measure μ which is nowhere-flat and cylinder-free. Suppose that $f, g : M \to \mathbb{C}$ and $h : 2M \to \mathbb{C}$ are measurable functions satisfying*

$$f(x)g(y) = h(x + y) \textit{ for } \mu^2\textit{-almost every } (x, y) \in M^2.$$

Suppose further that $f^{-1}(\{0\})$ *and* $g^{-1}(\{0\})$ *are* μ*-null. Then there exist unique* $v \in \mathbb{C}^d$ *and* $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ *such that* $f(x) = c_1 \exp(v \cdot x)$, $g(x) = c_2 \exp(v \cdot x)$ *for* μ*-almost every* $x \in M$.

The proof will also establish the analogous result for the additive Cauchy-Pexider functional equation.

Theorem 1.2. *Let* $d \geq 3$. *Let* $M \subset \mathbb{R}^d$ *be a connected hypersurface with induced measure* μ *which is nowhere-flat and cylinder-free. Suppose that* $f, g : M \to \mathbb{C}$ *and* $h : 2M \to \mathbb{C}$ *are measurable functions satisfying*

$$f(x) + g(y) = h(x+y) \text{ for } \mu^2\text{-almost every } (x,y) \in M^2.$$

Then there exist unique $v \in \mathbb{C}^d$ *and* $c_1, c_2 \in \mathbb{C}$ *such that* $f(x) = v \cdot x + c_1$, $g(x) = v \cdot x + c_2$ *for* μ*-almost every* $x \in M$.

In §4, the following approximate versions of these theorems are proved.

Theorem 1.3. *Let* $d \geq 3$. *Let* $M \subset \mathbb{R}^d$ *be a bounded hypersurface with finite induced measure* μ *which is nowhere-flat and cylinder-free. Given* $\epsilon > 0$ *there exists* $\delta > 0$ *with the following property. Whenever* $f, g : M \to \mathbb{C}$ *are measurable functions vanishing only on a* μ*-null set and* $h : 2M \to \mathbb{C}$ *is a measurable function satisfying*

$$\mu^2(\{(x,y) \in M^2 \mid |f(x)g(y)h(x+y)^{-1} - 1| > \delta\}) < \delta \tag{1.2}$$

it follows that there exist $c \in \mathbb{C}$ *and* $v \in \mathbb{C}^d$ *such that*

$$\mu(\{|f(x)(c\exp(v \cdot x))^{-1} - 1| > \epsilon\}) < \epsilon.$$

Theorem 1.4. *Let* $d \geq 3$. *Let* $M \subset \mathbb{R}^d$ *be a bounded hypersurface with finite induced measure* μ *which is nowhere-flat and cylinder-free. Given* $\epsilon > 0$ *there exists* $\delta > 0$ *with the following property. Whenever* $f, g : M \to \mathbb{C}$, $h : 2M \to \mathbb{C}$ *are measurable functions satisfying*

$$\mu^2(\{(x,y) \in M^2 \mid |f(x) + g(y) - h(x+y)| > \delta\}) < \delta \tag{1.3}$$

it follows that there exist $c \in \mathbb{C}$ *and* $v \in \mathbb{C}^d$ *such that*

$$\mu(\{|f(x) - (v \cdot x + c)| > \epsilon\}) < \epsilon.$$

The final sections deal with related restricted functional equations.

As in the two-dimensional example considered above, for a generic embedded curve $\Gamma \subset \mathbb{R}^d$ each $z \in d\Gamma$ can be written as

$$z = x_1 + \ldots + x_d$$

where $x_1, \ldots, x_d \in \Gamma$ and $x_1, \ldots, x_d$ are unique up to order. Thus the d-fold version of the Cauchy-Pexider equation restricted to curves will not force solutions to be exponential affine, in general. On the other hand, for a suitably non-degenerate curve, the $(d+1)$-fold version does; this is the content of the following theorems which are proved in §5.

Theorem 1.5. *Let $d \geq 2$. Let $\Gamma \subset \mathbb{R}^d$ be a connected submanifold of dimension 1 with induced measure γ. Suppose that no open subset of Γ lies in an affine hyperplane. Suppose that $f_0, \ldots, f_d : \Gamma \to \mathbb{C}$ and $F : (d+1)\Gamma \to \mathbb{C}$ are measurable functions satisfying*

$$f_0(x_0) \cdots f_d(x_d) = F(x_0 + \ldots + x_d) \text{ for } \gamma^{d+1}\text{-almost every } (x_0, \ldots, x_d) \in \Gamma^{d+1}.$$

Suppose further that for each $0 \leq j \leq d$, the set $f_j^{-1}(\{0\})$ is γ-null. Then there exist unique $v \in \mathbb{C}^d$ and $c_0, \ldots, c_d \in \mathbb{C}$ such that for each $0 \leq j \leq d$, $f_j(x) = c_j \exp(v \cdot x)$ for γ-almost every $x \in \Gamma$.

Theorem 1.6. *Let $d \geq 2$. Let $\Gamma \subset \mathbb{R}^d$ be a connected submanifold of dimension 1 with induced measure γ. Suppose that no open subset of Γ lies in an affine hyperplane. Suppose that $f_0, \ldots, f_d : \Gamma \to \mathbb{C}$ and $F : (d+1)\Gamma \to \mathbb{C}$ are measurable functions satisfying*

$$f_0(x_0) + \ldots + f_d(x_d) = F(x_0 + \ldots + x_d) \text{ for } \gamma^{d+1}\text{-almost every } (x_0, \ldots, x_d) \in \Gamma^{d+1}.$$

Then there exist unique $v \in \mathbb{C}^d$ and $c_0, \ldots, c_d \in \mathbb{C}$ such that for each $0 \leq j \leq d$, $f_j(x) = v \cdot x + c_j$ for γ-almost every $x \in \Gamma$.

In §6, the two-fold product in the left-hand side of (1.1) is replaced by a product of three functions; by combining an adaptation of the strategy in §2 with Theorems 1.5, 1.6, we establish the following theorems for hypersurfaces.

Theorem 1.7. *Let $d \geq 2$. Let $M \subset \mathbb{R}^d$ be a connected hypersurface with induced measure μ which is nowhere-flat. Suppose that $f_1, f_2, f_3 : M \to \mathbb{C}$ and $F : 3M \to \mathbb{C}$ are measurable functions satisfying*

$$f_1(x_1)f_2(x_2)f_3(x_3) = F(x_1 + x_2 + x_3) \text{ for } \mu^3\text{-almost every } (x_1, x_2, x_3) \in M^3.$$

Suppose further that $f_j^{-1}(\{0\})$ is μ-null for $1 \leq j \leq 3$. Then there exist unique $v \in \mathbb{C}^d$ and $c_1, c_2, c_3 \in \mathbb{C}$ such that for $j = 1, 2, 3$, $f_j(x) = c_j \exp(v \cdot x)$ for μ-almost every $x \in M$.

Theorem 1.8. *Let $d \geq 2$. Let $M \subset \mathbb{R}^d$ be a connected hypersurface with induced measure μ which is nowhere-flat. Suppose that $f_1, f_2, f_3 : M \to \mathbb{C}$ and $F : 3M \to \mathbb{C}$ are measurable functions satisfying*

$$f_1(x_1) + f_2(x_2) + f_3(x_3) = F(x_1 + x_2 + x_3) \text{ for } \mu^3\text{-almost every } (x_1, x_2, x_3) \in M^3.$$

Then there exist unique $v \in \mathbb{C}^d$ and $c_1, c_2, c_3 \in \mathbb{C}$ such that for $1 \leq j \leq 3$, $f_j(x) = v \cdot x + c_j$ for μ-almost every $x \in M$.

Other than in the aforementioned [2], these functional equations have arisen in Foschi [4] (for four specific submanifolds M). The proof provided in this paper presents an alternative approach to the methods in [4], one which is less reliant on the rigid geometry of particular M.

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2. A General Strategy

Define λ^n to be n-dimensional Lebesgue measure on $\mathbb{R}^n$ and $\lambda = \lambda^1$. By a slight abuse of notation, we will also write λ^n for the induced measure on an n-dimensional linear subspace of $\mathbb{R}^d$.

A starting point for our proofs of Theorems 1.1 and 1.3 will be the following approximate version for the solution to the Cauchy-Pexider functional equation on open balls of $\mathbb{R}^d$ which is proved in Christ [3].

Lemma 2.1. *Given $\epsilon > 0$ there exists $\delta > 0$ depending on the dimension d with the following property. Whenever $B \subset \mathbb{R}^d$ is a non-empty ball and $f : B \to \mathbb{C}$, $g : B \to \mathbb{C}$, $h : 2B \to \mathbb{C}$ are measurable functions vanishing only on λ^d-null sets and satisfying*

$$\lambda^{2d}(\{(x, y) \in B^2 \mid |f(x)g(y)h(x + y)^{-1} - 1| > \delta\}) < \lambda^d(B)^2\delta,$$

it follows that there exist $c \in \mathbb{C}\backslash\{0\}$ and $v \in \mathbb{C}^d$ such that

$$\lambda^d(\{x \in B \mid |f(x)(c \exp(v \cdot x))^{-1} - 1| > \epsilon\}) < \lambda^d(B)\epsilon.$$

If, in addition, $f(x)g(y) = h(x + y)$ for λ^{2d}-a.e. $(x, y) \in B^2$ then there exist $c \in \mathbb{C}\backslash\{0\}$ and $v \in \mathbb{C}^d$ such that $f(x) = c \exp(v \cdot x)$ for λ^d-a.e. $x \in B$.

The next lemma will allow us to work locally.

Lemma 2.2. *Suppose that the hypersurface M with induced measure μ is nowhere-flat. Let U, V be open subsets of M with non-empty intersection. Suppose that the function $f : U \cup V \to \mathbb{C}$ satisfies $f(x) = c \exp(v \cdot x)$ for μ-a.e. $x \in U$ and $f(x) = c_1 \exp(v_1 \cdot x)$ for μ-a.e. $x \in V$ where $v, v_1 \in \mathbb{C}^d$ and $c, c_1 \in \mathbb{C}\backslash\{0\}$. Then $v = v_1$ and $c = c_1$.*

Proof. For μ-almost every $x \in U \cap V$,

$$c \exp(v \cdot x) = f(x) = c_1 \exp(v_1 \cdot x).$$

Thus for all such x,

$$(\text{Re } v - \text{Re } v_1) \cdot x = a \tag{2.1}$$

$$(\text{Im } v - \text{Im } v_1) \cdot x = b + 2\pi n(x) \tag{2.2}$$

where $a, b \in \mathbb{R}$ satisfy $e^{a+ib} = c_1 c^{-1}$ and $n : U \cap V \to \mathbb{Z}$ is some function. Since M is nowhere-flat, no open subset of M can lie on an affine hyperplane. Thus $v = v_1$ so also $c_1 c^{-1} = 1$. □

In particular, the lemma implies the stated uniqueness in Theorem 1.1. Furthermore, it will suffice to prove the theorem for some open cover of M. Thus it will be no loss of generality to prove Theorem 1.1 for a connected, orientable, bounded hypersurface M with μ finite. In particular, this means that the Gauss normal map $\mathcal{G}$ is globally defined on M. Define also the map $\overline{\mathcal{G}} : M \to \overline{\mathbb{S}^{d-1}}$, where $\overline{\mathbb{S}^{d-1}}$ is the unit sphere with antipodal points identified, which maps $x \in M$ to the class of $\mathcal{G}(x)$. We will identify the tangent space T_xM with the linear hyperplane $\text{span}\{\mathcal{G}(x)\}^\perp \subset \mathbb{R}^d$ in the canonical way.

In this section, the strategy used in [2] for $\mathbb{S}^2 \subset \mathbb{R}^3$ will be generalised to hypersurfaces. The proof of Theorem 1.1 will be completed in the subsequent section.

Let $d \geq 3$ and write points in $(\mathbb{R}^d)^4 \times (\mathbb{R}^d)^4$ as (x, y) where $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ with $x_j, y_j \in \mathbb{R}^d$ for $1 \leq j \leq 4$. Let Π be the hyperplane in $(\mathbb{R}^d)^4 \times (\mathbb{R}^d)^4$ defined by $x_1 + y_2 = x_3 + y_4$ and $y_1 + x_2 = y_3 + x_4$. Let $\mathcal{P}_M = (M^4 \times M^4) \cap \Pi$. By a *smooth point* of $\mathcal{P}_M$ we mean a point where $M^4 \times M^4$ intersects Π transversally. Write $\mathcal{S}_M$ for the set of smooth points of $\mathcal{P}_M$. Then $\mathcal{S}_M$ is a submanifold of $\mathbb{R}^{8d}$ of dimension $8(d-1) + 6d - 8d = 6d - 8$. Write σ for the volume measure on $\mathcal{S}_M$ associated to the induced Riemannian structure.

Consider the $3d$-dimensional hyperplane $\Lambda \subset (\mathbb{R}^d)^4$ of points $w = (w_1, w_2, w_3, w_4)$ satisfying $w_1 + w_2 = w_3 + w_4$. The linear addition map $(\mathbb{R}^d)^4 \times (\mathbb{R}^d)^4 \to (\mathbb{R}^d)^4$, $(x, y) \mapsto x + y$ restricts to a smooth map $\pi_M : \mathcal{S}_M \to \Lambda$. Write $\mathcal{R}_M$ for the set of *regular points* of π_M, that is the points of $\mathcal{S}_M$ where π_M is a submersion. Then $\mathcal{R}_M$ is an open subset of $\mathcal{S}_M$.

Lemma 2.3. *Let $(x, y) \in \mathcal{P}_M$. If*

$$\overline{\mathcal{G}}(x_1) \neq \overline{\mathcal{G}}(y_2) \text{ and } \overline{\mathcal{G}}(x_2) \neq \overline{\mathcal{G}}(y_1) \tag{2.3}$$

then (x, y) is a smooth point of $\mathcal{P}_M$.

If, in addition,

$$\overline{\mathcal{G}}(x_j) \neq \overline{\mathcal{G}}(y_j) \text{ for } 1 \leq j \leq 4 \tag{2.4}$$

and

$$\bigcap_{j=1}^{4} \operatorname{span}\{\mathcal{G}(x_j), \mathcal{G}(y_j)\} = \{0\} \tag{2.5}$$

then (x, y) is a regular point of π_M.

Proof. The point $(x, y) \in \mathcal{P}_M$ is smooth if and only if

$$T_{x_1}M + T_{y_2}M - T_{x_3}M - T_{y_4}M = T_{y_1}M + T_{x_2}M - T_{y_3}M - T_{x_4}M = \mathbb{R}^d.$$

Thus condition (2.3) is sufficient.

For $(x, y) \in \mathcal{S}_M$, we identify $T_{(x,y)}\mathcal{S}_M$ with $(\prod_{j=1}^4 T_{x_j}M \times \prod_{j=1}^4 T_{y_j}M) \cap \Pi$ and $T_{x+y}\Lambda$ with Λ in the canonical way. The differential $(d\pi_M)_{(x,y)} : T_{(x,y)}\mathcal{S}_M \to T_{x+y}\Lambda$ is then given by

$$(d\pi_M)_{(x,y)}(u, v) = u + v$$

for $(u, v) = (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)$ with $u_j \in T_{x_j}M$, $v_j \in T_{y_j}M$ for $1 \leq j \leq 4$ and $u_1 + v_2 = u_3 + v_4$, $v_1 + u_2 = v_3 + u_4$. Thus for $(d\pi_M)_{(x,y)}$ to be surjective we certainly need $T_{x_j}M + T_{y_j}M = \mathbb{R}^d$ for each $1 \leq j \leq 4$. This condition is equivalent to (2.4).

Let $w \in \Lambda$ and assume that (2.4) and (2.5) hold. It suffices to show that there exist $u_j \in T_{x_j}M$, $v_j \in T_{y_j}M$ for $1 \leq j \leq 4$ such that $u_1 + v_2 = u_3 + v_4$ and $u_j + v_j = w_j$ for $1 \leq j \leq 4$; we then also have

$$v_1 + u_2 = (w_1 - u_1) + (w_2 - v_2) = (w_3 + w_4) - (u_3 + v_4) = v_3 + u_4.$$

From the equation $u_j + v_j = w_j$, we can vary u_j freely over a certain translate of the linear space $T_{x_j}M \cap T_{y_j}M = \operatorname{span}\{\mathcal{G}(x_j), \mathcal{G}(y_j)\}^{\perp}$ and then $v_j = w_j - u_j$ is determined.

Alternatively we can vary v_j freely over a certain translate of $\operatorname{span}\{\mathcal{G}(x_j), \mathcal{G}(y_j)\}^\perp$ and then u_j is determined. As u_1, v_2, u_3, v_4 vary freely over these affine spaces, condition (2.5) implies that $u_1 + v_2 - u_3 - v_4$ varies over all of $\mathbb{R}^d$. In particular, there exists a choice such that $u_1 + v_2 = u_3 + v_4$. □

For $S \subset M^2$, let $\mathcal{S}_M(S) = \{(x, y) \in \mathcal{S}_M \,|\, (x_i, y_j) \in S \text{ for all } 1 \leq i, j \leq 4\}$. The above setup is motivated by the following observation.

Lemma 2.4. *Let $f : M \to \mathbb{C}$, $g : M \to \mathbb{C}$ and $h : 2M \to \mathbb{C}$. Suppose that $f(x)g(y) = h(x + y)$ for all $(x, y) \in S \subset M^2$. Whenever $z \in \Lambda$ is in the image of $\mathcal{S}_M(S)$ under π_M,*

$$h(z_1)h(z_2) = h(z_3)h(z_4).$$

Proof. Consider $z = \pi_M(x, y) = x + y$ for $(x, y) \in \mathcal{S}_M(S)$. By expanding using (1.1) and regrouping,

$$h(z_1)h(z_2) = f(x_1)g(y_1)f(x_2)g(y_2) = h(x_1 + y_2)h(y_1 + x_2)$$

and similarly

$$h(z_3)h(z_4) = f(x_3)g(y_3)f(x_4)g(y_4) = h(x_3 + y_4)h(y_3 + x_4).$$

But $x_1 + y_2 = x_3 + y_4$ and $y_1 + x_2 = y_3 + x_4$. □

Define $R_M = \{x + y \mid x, y \in M,\ \overline{\mathcal{G}}(x) \neq \overline{\mathcal{G}}(y)\}$.

Lemma 2.5. *The set R_M is an open subset of $\mathbb{R}^d$. If M is nowhere-flat then R_M is dense in $2M$.*

Proof. The addition map $\alpha : M^2 \to \mathbb{R}^d$, $(x, y) \mapsto x + y$ is a submersion at (x, y) if and only if $T_xM + T_yM = \mathbb{R}^d$. So R_M is the image of the regular points of α. If M is nowhere-flat then $\mathcal{G}$ is not constant on any open subset of M. □

From the proof of Lemma 2.3, it follows that condition (2.4) is also necessary for (x, y) to be a regular point of π_M. Thus we can define a submersion $\gamma_M : \mathcal{R}_M \to R_M$ given by $(x, y) \mapsto x_1 + y_1$. The following proposition describes the main property of γ_M we will need; the proof will be given in the next section.

Proposition 2.6. *Let M be nowhere-flat and cylinder-free. Then the submersion $\gamma_M : \mathcal{R}_M \to R_M$ is surjective.*

If $\pi : P \to N$ is smooth and ρ is a measure on P, write $\pi_*(\rho)$ for the pushforward of ρ under π, that is the measure on N defined by $\pi_*(\rho)(E) = \rho(\pi^{-1}(E))$. The following fact will be useful; see, for example, [5] for a proof.

Lemma 2.7. *Suppose that $\pi : P \to N$ is a surjective submersion between bounded submanifolds $P \subset \mathbb{R}^p$ and $N \subset \mathbb{R}^n$. Let ρ and ν denote the measures associated to the induced Riemannian structures on P and N respectively. Then ν and $\pi_*(\rho)$ are mutually absolutely continuous.*

Observe that for each $1 \leq i, j \leq 4$, the smooth map $\mathcal{R}_M \to M^2$ given by $(x, y) \mapsto (x_i, y_j)$ is a submersion. Thus, by Lemma 2.7, if $S \subset M^2$ is μ^2-full then

$$\sigma(\mathcal{R}_M \backslash \mathcal{R}_M \cap \mathcal{S}_M(S)) = 0. \tag{2.6}$$

Let M be nowhere-flat and cylinder-free. The addition map $M^2 \to \mathbb{R}^d$ restricts to a surjective submersion $\alpha_M : \mathcal{M} \to R_M$ where $\mathcal{M}$ is the open subset of M^2 consisting of points (x, y) such that $\overline{\mathcal{G}}(x) \neq \overline{\mathcal{G}}(y)$. The measure $\mu^2|_{\mathcal{M}}$ coincides with the induced measure on $\mathcal{M}$ as a submanifold of $(\mathbb{R}^d)^2$. Since $\mu^2(M^2 \backslash \mathcal{M}) = 0$, Lemma 2.7 implies that

$$\lambda^d|_{R_M} \text{ and } (\mu * \mu)|_{R_M} \text{ are mutually absolutely continuous} \tag{2.7}$$

where $\mu * \mu$ denotes the measure given by $\mu * \mu(E) = \mu^2(\{(x, y) \in M^2 \mid x + y \in E\})$.

Suppose that $f : M \to \mathbb{C}$, $g : M \to \mathbb{C}$ are measurable functions which only vanish on sets of μ-measure zero and $h : 2M \to \mathbb{C}$ is measurable. By (2.7), h vanishes only on a set of λ^d-measure zero.

Suppose that (1.1) is satisfied for some $S \subset M^2$ with $\mu^2(M^2 \backslash S) = 0$. Let $z_1 \in R_M$. By Proposition 2.6, there exists $(x, y) \in \mathcal{R}_M$ such that $z_1 = \gamma_M(x, y)$. Write $z = (z_1, z_2, z_3, z_4) = \pi_M(x, y) \in \Lambda$. Since π_M is a submersion at (x, y), Lemma 2.4 together with (2.6) imply that for λ^{3d}-a.e. $w \in \Lambda$ in an open neighbourhood V of z,

$$h(w_1)h(w_2) = h(w_3)h(w_4).$$

By shrinking V if necessary, we may assume that $V = (T_1 \times T_2 \times T_3 \times T_4) \cap \Lambda$ where each T_j is an open connected subset of R_M, T_1 is a ball with center at z_1, T_2 is a translate of T_1 and the projection $\Lambda \to (\mathbb{R}^d)^2$ given by $(w_1, w_2, w_3, w_4) \mapsto (w_1, w_2)$ maps V onto $T_1 \times T_2$. Consider the set $\mathcal{T}$ of all tuples $(w_1, w_2, w'_1, w'_2, w_3, w_4)$ where $w_1, w'_1 \in T_1$, $w_2, w'_2 \in T_2$, $w_3 \in T_3$, $w_4 \in T_4$ and $w_1 + w_2 = w'_1 + w'_2 = w_3 + w_4$. Then $\mathcal{T}$ is an open subset of a $4d$-dimensional linear space. The projections given by $(w_1, w_2, w'_1, w'_2, w_3, w_4) \mapsto (w_1, w_2, w_3, w_4)$ and $(w_1, w_2, w'_1, w'_2, w_3, w_4) \mapsto (w'_1, w'_2, w_3, w_4)$ are surjective submersions $\mathcal{T} \to V$. Therefore, by Lemma 2.7, for λ^{3d}-a.e. $(w_1, w_2, w'_1, w'_2) \in (T_1 \times T_2 \times T_1 \times T_2) \cap \Lambda$,

$$h(w_1)h(w_2) = h(w'_1)h(w'_2).$$

By setting $H(u)$ to be the average of $h(w_1)h(w_2)$ over pairs $(w_1, w_2) \in T_1 \times T_2$ satisfying $w_1 + w_2 = u$, we obtain a measurable function $H : T_1 + T_2 \to \mathbb{C}$ such that $H(w_1 + w_2) = h(w_1)h(w_2)$ for λ^{2d}-a.e. $(w_1, w_2) \in T_1 \times T_2$. Applying Lemma 2.1, we deduce that there exist $c \neq 0$ and $v \in \mathbb{C}^d$ such that $h(w) = c \exp(v \cdot w)$ for λ^d-a.e. $w \in T_1$.

Thus we have shown that for each $z \in R_M$, there exist an open connected neighbourhood $B_z \subset R_M$ of z, $v_z \in \mathbb{C}^d$ and $c_z \neq 0$ such that $h(w) = c_z \exp(v_z \cdot w)$ for λ^d-a.e. $w \in B_z$. Define an equivalence relation $\sim$ on R_M by declaring $w \sim z$ whenever $v_w = v_z$. Observe that if $z_0 \in R_M$ then $z \sim z_0$ for any $z \in B_{z_0}$. Therefore, the equivalence relation partitions R_M into a collection of open sets $\{R_j\}_{j \in J}$ such that each $z \in R_j$ has a neighbourhood on which h is of the form $h(z) = c \exp(v_j \cdot z)$, up to a null set, for some c depending on the neighbourhood.

For $j \in J$, consider the set M_j of all $x \in M$ for which there exists some $y \in M$ such that $x + y \in R_j$. Since M is nowhere-flat, it follows that the sets $\{M_j\}_{j \in J}$ are an open cover for M. Furthermore, the sets are disjoint. Indeed, suppose that the open set $M_i \cap M_j$ is non-empty. Then there are $y_1, y_2 \in M$ and an open subset U of M such that $U + y_1 \in R_i$, $U + y_2 \in R_j$, $g(y_1), g(y_2) \neq 0$ and the restricted Cauchy-Pexider

equation (1.1) is satisfied for $S = U_1 \times \{y_1, y_2\}$ where U_1 is a subset of U with full μ-measure. By shrinking U if necessary, for μ-a.e. $x \in U$,

$$g(y_1)^{-1} c \exp(v_i \cdot x) = f(x) = g(y_2)^{-1} c' \exp(v_j \cdot x)$$

for some constants $c, c' \neq 0$. By Lemma 2.2, this implies that $v_i = v_j$ so $M_i = M_j$.

Since M is connected, it follows that $M_j = M$ for some $j \in J$. This means that each $x \in M$ has an open neighbourhood U in M on which f is of the form $f(x) = c_U \exp(v_j \cdot x)$ except for a μ-null set for some constant $c_U \neq 0$. Appealing to the assumption that M is connected once again, it follows that there is a constant $c_1 \neq 0$ such that $f(x) = c_1 \exp(v_j \cdot x)$ for μ-almost every all $x \in M$. Similarly, $g(y) = c_2(v_j \cdot y)$ for μ-almost every $y \in M$.

Thus the proof of Theorem 1.1 will be complete once we establish Proposition 2.6. Lemmata 2.2 and 2.4 have straighforward analogues for the additive Cauchy-Pexider functional equation with similar proofs; the details are omitted. By combining these with the additive analogue of Lemma 2.1 (see [3]) and running through the argument above, we obtain Theorem 1.2. □

3. The Proof of Proposition 2.6

Definition 3.1. A *planar circle* in $\mathbb{S}^{d-1}$ is the intersection in $\mathbb{R}^d$ of a two-dimensional linear subspace with $\mathbb{S}^{d-1}$.

Lemma 3.1. *Suppose that $M \subset \mathbb{R}^d$ is a nowhere-flat hypersurface. Then M is cylinder-free if and only if whenever U is an open subset of M, the set $\mathcal{G}(U) \subset \mathbb{S}^{d-1}$ is not contained in a planar circle.*

Proof. Suppose that $\mathcal{G}(U)$ is contained in the planar circle $\Pi \cap \mathbb{S}^{d-1}$ where $\Pi \subset \mathbb{R}^d$ is a two-dimensional plane. Since M is nowhere-flat, for each $x \in U$ there is exactly one non-zero principal curvature with principal direction p_x, say, and $\Pi = \operatorname{span}\{\mathcal{G}(x), p_x\}$. Each direction $v \in \Pi^\perp$ then defines a unit vector field of constant direction v on U. Thus there is an integral curve which is a line segment in the direction v through each $x \in U$. Choose $x_0 \in U$ and consider the smooth curve $\Gamma = (x_0 + \Pi) \cap U$ through x_0. There is a line segment in each direction $v \in \Pi^\perp$ at each point $x \in \Gamma$ which lies entirely in U and moreover these line segments can be chosen to vary smoothly with x and v. Thus, U contains a cylinder.

The converse follows from the observation that $\mathcal{G}(U)$ is contained in a planar circle whenever U is a cylinder. □

Let M be a connected, nowhere-flat, cylinder-free hypersurface. For each $w \in R_M$, the submanifolds M and $w - M$ of $\mathbb{R}^d$ intersect transversally at a point x if and only if $\overline{\mathcal{G}}(x) \neq \overline{\mathcal{G}}(w - x)$. Write $M_w \subset M \cap (w - M)$ for the set of points where M meets $w - M$ transversally. Then M_w is a non-empty submanifold of $\mathbb{R}^d$ of codimension 2. Define $D_M = \{w \in R_M \mid \mathcal{G}(M_w) \text{ is not contained in a planar circle}\}$.

Lemma 3.2. *Let M be nowhere-flat and cylinder-free. Then D_M is open and dense in R_M.*

Proof. Let $w \in D_M$. Then there exist $x_1, x_2, x_3 \in M_w$ such that $\mathcal{G}(x_1), \mathcal{G}(x_2), \mathcal{G}(x_3)$ are not contained in a planar circle. Recall the surjective submersion $\alpha_M : \mathcal{M} \to R_M$, $(x, y) \mapsto x+y$. Let $\theta_j : U \to \mathcal{M}$ for $j = 1, 2, 3$ be smooth local sections of α_M defined on some open $U \subset R_M$ containing w such that $\theta_j(w) = (x_j, w - x_j)$. Let $p : \mathcal{M} \to M$ be the restriction to $\mathcal{M}$ of the projection onto the first coordinate. For all z in a small neighbourhood of w in U, the points $\mathcal{G}(p \circ \theta_1(z)), \mathcal{G}(p \circ \theta_2(z)), \mathcal{G}(p \circ \theta_3(z))$ will also not be contained in a planar circle. Thus D_M is open.

Suppose for contradiction that D_M is not dense. Let $S \subset R_M$ be non-empty and open such that for all $w \in S$, the set $\mathcal{G}(M_w)$ is contained in the planar circle $\Pi_w \cap \mathbb{S}^{d-1}$ where Π_w is a two-dimensional plane. For any $x \in M_w$, $\overline{\mathcal{G}}(x) \neq \overline{\mathcal{G}}(w - x)$ so $\Pi_w =$ span$\{\mathcal{G}(x), \mathcal{G}(w - x)\}$ and the tangent space

$$T_x M_w = T_x M \cap T_{w-x} M = \mathrm{span}\{\mathcal{G}(x), \mathcal{G}(w - x)\}^\perp = \Pi_w^\perp.$$

Fix $w_0 \in S$ and $x_0 \in M_{w_0}$. Set $y_0 = w_0 - x_0 \in M$. In particular, this means that $\overline{\mathcal{G}}(y_0) = \overline{\mathcal{G}}(w_0 - x_0) \neq \overline{\mathcal{G}}(x_0)$ and there exists a small open neighbourhood B of y_0 in M, such that for all $y \in B$, $\overline{\mathcal{G}}(y) \neq \overline{\mathcal{G}}(x_0)$ and $w(y) = x_0 + y \in S$. Since $\mathcal{G}(M_{w(y)}) \subset \Pi_{w(y)}$, it follows that $d\mathcal{G}_{x_0}(T_{x_0} M_{w(y)}) \subset \Pi_{w(y)}$. Thus for any unit vector $v \in \Pi_{w(y)}^\perp = \mathrm{span}\{\mathcal{G}(x_0), \mathcal{G}(y)\}^\perp$,

$$d\mathcal{G}_{x_0}(v) \cdot v = 0. \tag{3.1}$$

Let $\{q_1, \ldots, q_{d-1}\}$ be a set of principal directions at x_0 with corresponding principal curvatures $\kappa_1, \ldots, \kappa_{d-1}$. So $\{q_1, \ldots, q_{d-1}\}$ is a basis for $T_{x_0} M$ and $\{q_1, \ldots, q_{d-1}, \mathcal{G}(x_0)\}$ is an orthonormal basis for $\mathbb{R}^d$. Furthermore, the set $\{q_1, \ldots, q_{d-1}\}$ is a basis of eigenvectors for the linear map $d\mathcal{G}_{x_0}$ with eigenvalues $-\kappa_1, \ldots, -\kappa_{d-1}$.

Define $\overline{\mathbf{r}}(y) = (q_1 \cdot \mathcal{G}(y), \ldots, q_{d-1} \cdot \mathcal{G}(y))$, $\mathbf{v} = (v_1, \ldots, v_{d-1})$ and $\mathbf{k} = (\kappa_1, \ldots, \kappa_{d-1})$ viewed as elements of $\mathbb{R}^{d-1}$. Since $\overline{\mathcal{G}}(y) \neq \overline{\mathcal{G}}(x_0)$, it follows that $\overline{\mathbf{r}}(y) \neq 0$ and we can define $\mathbf{r}(y) = \overline{\mathbf{r}}(y)/\|\overline{\mathbf{r}}(y)\|$. Also define

$$H_{\mathbf{k}} = \{\mathbf{u} \in \mathbb{R}^{d-1} \,|\, \sum_{j=1}^{d-1} \kappa_j u_j^2 = 0\}.$$

Then, equation (3.1) is equivalent to the assertion that whenever $\mathbf{v} \in \mathbb{S}^{d-2}$ and $\mathbf{v} \cdot \mathbf{r}(y) = 0$ it follows that $\mathbf{v} \in H_{\mathbf{k}}$. Since M is nowhere-flat, $\mathbf{k} \neq 0$. Suppose without loss of generality that $\kappa_{d-1} \neq 0$. Let $\Phi = \{\mathbf{u} \in \mathbb{R}^{d-1} \,|\, u_{d-1} \neq 0\}$. Then the set $H_{\mathbf{k}} \cap \mathbb{S}^{d-2} \cap \Phi$ is a $(d-3)$-dimensional submanifold of $\mathbb{R}^{d-1}$. Observe that $\mathbf{r}(y)$ varies smoothly with $y \in B$ on $\mathbb{S}^{d-2}$. If it is not constant, then

$$\bigcup_{y \in B} \mathrm{span}\{\mathbf{r}(y)\}^\perp \cap \mathbb{S}^{d-2}$$

contains an open subset of $\mathbb{S}^{d-2}$. For every $y \in B$, the $(d-3)$-dimensional submanifold span$\{\mathbf{r}(y)\}^\perp \cap \mathbb{S}^{d-2}$ is a subset of $H_{\mathbf{k}} \cap \mathbb{S}^{d-2}$, thus $H_{\mathbf{k}} \cap \mathbb{S}^{d-2}$ contains an open subset of $\mathbb{S}^{d-2}$. Since $\Phi \cap \mathbb{S}^{d-2}$ is a dense open subset of $\mathbb{S}^{d-2}$, this implies that an open subset of a $(d-2)$-dimensional manifold is contained in a $(d-3)$-dimensional manifold; this is a contradiction. Therefore $\mathbf{r}(y) = (r_1, \ldots, r_{d-1})$ is constant.

Thus for all $y \in B$,

$$\mathcal{G}(y) \in \operatorname{span}\{\mathcal{G}(x_0), \sum_{j=1}^{d-1} r_j q_j\}.$$

By Lemma 3.1, this contradicts M being cylinder-free. □

We now prove Proposition 2.6. Let $w \in R_M$. By Lemma 2.3, it suffices to show that there exists $(x, y) \in \mathcal{P}_M$ satisfying (2.3), (2.4) and (2.5).

The set $M + M_w$ contains a subset which is open in $\mathbb{R}^d$. Indeed, consider the smooth map $\tau : M \times M_w \to \mathbb{R}^d$, $(x, s) \mapsto x + s$. Then τ is a submersion at (x, s) if and only if $\mathcal{G}(x)$ does not lie in the span of $\mathcal{G}(s)$ and $\mathcal{G}(w-s)$. Since M is cylinder-free the set $\mathcal{G}(M)$ does not lie in any planar circle, thus given any $s \in M_w$ there exists $x \in M$ such that (x, s) is a regular point of τ. Therefore the image of τ must contain an open set.

By Lemma 3.2, there thus exist $x_1, y_1, y_2 \in M$ such that $x_1 + y_2 \in D_M$, $x_1 + y_1 = w$ and $\overline{\mathcal{G}}(x_1) \neq \overline{\mathcal{G}}(y_1)$.Since M is nowhere-flat, y_2 may be perturbed in a small M-open set if necessary so that $\overline{\mathcal{G}}(y_2) \neq \overline{\mathcal{G}}(x_1)$.

Since M is cylinder-free, there exists $x_2 \in M$ such that

$$\mathcal{G}(x_2) \notin \operatorname{span}\{\mathcal{G}(x_1), \mathcal{G}(y_1)\}. \tag{3.2}$$

By perturbing x_2 if necessary we may assume in addition that

$$\overline{\mathcal{G}}(x_2) \neq \overline{\mathcal{G}}(y_2). \tag{3.3}$$

The points y_3 and x_4 will be chosen to be close to y_1 and x_2 respectively. To this end, set $y_3 = y_1$ and $x_4 = x_1$ for the moment. Since $x_1 + y_2 \in D_M$, there exists y_4 such that $y_4 \in M \cap (x_1 + y_2 - M)$ and

$$\mathcal{G}(y_4) \notin \operatorname{span}\{\mathcal{G}(x_2), \mathcal{G}(y_2)\}. \tag{3.4}$$

Let $x_3 = x_1 + y_2 - y_4 \in M$.

Define $\Delta = \Delta(x_1, x_2, y_1, y_2) = \operatorname{span}\{\mathcal{G}(x_1), \mathcal{G}(y_1)\} \cap \operatorname{span}\{\mathcal{G}(x_2), \mathcal{G}(y_2)\}$. The choice of x_1, y_1, x_2, y_2 implies that $\mathcal{G}(x_2) \notin \Delta$ and $\dim \Delta \leq 1$. Then, by (3.2), (3.3), (3.4), it follows that

$$\mathcal{G}(x_4) \notin \operatorname{span}\{\mathcal{G}(y_4)\} + \Delta \tag{3.5}$$

so $\overline{\mathcal{G}}(x_4) \neq \overline{\mathcal{G}}(y_4)$ and $\Delta \cap \operatorname{span}\{\mathcal{G}(x_4), \mathcal{G}(y_4)\} = \{0\}$. Therefore (2.3) and (2.5) are satisfied. Moreover, (2.4) is also satisfied except perhaps for the statement $\overline{\mathcal{G}}(x_3) \neq \overline{\mathcal{G}}(y_3)$.

Suppose that $\overline{\mathcal{G}}(x_3) = \overline{\mathcal{G}}(y_3)$ for this choice of (x, y). Keep x_1, x_3, y_1, y_2, y_3 and y_4 fixed. Then x_2 may be perturbed freely in an open subset B of M, with x_4 set to x_2 for each choice, while preserving the conditions (3.2), (3.3) and (3.4). Furthermore, for each choice of $x_2 \in B$, we may perturb y_3 in a small open subset of $M_{y_1+x_2}$ and redefine $x_4 = y_1 + x_2 - y_3$ while still preserving condition (3.5). The next lemma completes the proof of Proposition 2.6.

Lemma 3.3. *Let $B \subset M$ be non-empty and open. Suppose that $y \in M$ satisfies $\overline{\mathcal{G}}(y) \notin \overline{\mathcal{G}}(B)$. Then, for any open neighbourhood $U \subset M$ of y, there exists $x \in B$ and $y' \in M_{x+y} \cap U$ such that $\overline{\mathcal{G}}(y') \neq \overline{\mathcal{G}}(y)$.*

Proof. Suppose not. Then for each $x \in B$, the Gauss normal map $\mathcal{G}$ is constant along M_{x+y} near y. Thus, the $(d-2)$-dimensional tangent plane $T_y M_{x+y}$ must lie entirely in the span of the principal directions at $y \in M$ corresponding to zero principal curvatures. Since M is nowhere-flat, there exists a principal direction p at y with non-zero principal curvature. Since $T_y M_{x+y} = T_y M \cap T_y(x+y-M) = T_y M \cap T_x M$, this means that p lies in the span of $\mathcal{G}(y)$ and $\mathcal{G}(x)$. Since this is true for all $x \in B$ and $\mathcal{G}(y) \cdot p = 0$ it follows that $\mathcal{G}(B)$ lies entirely in the span of p and $\mathcal{G}(y)$. By Lemma 3.1, this contradicts M being cylinder-free. $\square$

4. Nearly-Multiplicative Functions on Hypersurfaces

In this section we establish Theorem 1.3, which is an approximate version of the main result Theorem 1.1. By replacing multiplication with addition where appropriate, the same argument also establishes the approximate version of Theorem 1.2.

If $F : (0, \delta_0) \to \mathbb{C}$ is a function with domain the open interval $(0, \delta_0)$ for some $\delta_0 > 0$ and $\tau_1, \ldots, \tau_k$ is a list of parameters, we will use the Landau notation

$$F(\delta) = o_{\tau_1,\ldots,\tau_k}(1)$$

to mean that $F(\delta) \to 0$ as $\delta \to 0$ in a way which only depends on $\tau_1, \ldots, \tau_k, M$—that is, there exists some $0 < \delta_1 < \delta_0$ and a function $\nu : (0, \delta_1) \to (0, \infty)$ depending only on $\tau_1, \ldots, \tau_k$ and M satisfying $\nu(\delta) \to 0$ as $\delta \to 0$ such that for all $0 < \delta < \delta_1$

$$|F(\delta)| \leq \nu(\delta).$$

We will write $F(\delta) = G(\delta) + o_{\tau_1,\ldots,\tau_k}(1)$ to mean $F(\delta) - G(\delta) = o_{\tau_1,\ldots,\tau_k}(1)$.

Let M be as in the statement of Theorem 1.3. Suppose that $f, g : M \to \mathbb{C}$ are measurable functions vanishing only on a μ-null set and $h : 2M \to \mathbb{C}$ is measurable. For each $\delta > 0$ define

$$E_\delta = \{(x, y) \in M^2 \mid |f(x)g(y)h(x+y)^{-1} - 1| > \delta\}.$$

By closely following the discussion in §2 which relies on Proposition 2.6, replacing null sets with small sets where appropriate, we obtain the following proposition; the precise details are omitted.

Proposition 4.1. *For all $z \in R_M$ there exist a non-empty open ball $T_z \subset \mathbb{R}^d$ with center z and a translate B_z of T_z with the following property. If $\mu^2(E_\delta) < \delta$ then there exist a measurable function $H_z^\delta : T_z + B_z \to \mathbb{C}$ and a set $\mathcal{B}_z^\delta \subset T_z \times B_z$ such that*

$$h(w_1)h(w_2)H_z^\delta(w_1 + w_2)^{-1} = 1 + o(1)$$

for all $(w_1, w_2) \in \mathcal{B}_z^\delta$ and

$$\lambda^{2d}(\mathcal{B}_z^\delta) = o_z(1).$$

By Lemma 2.1, the proposition implies that for each $z \in R_M$ if $\mu^2(E_\delta) < \delta$ then there exist $c_z^\delta \in \mathbb{C}\backslash\{0\}$ and $v_z^\delta \in \mathbb{C}^d$ satisfying

$$h(w)(c_z^\delta \exp(v_z^\delta \cdot w))^{-1} = 1 + o_z(1)$$

for all $w \in T_z$ except for a set of λ^d-measure $o_z(1)$. Define

$$\mathcal{N}_z = \{x' \in M \mid \text{there exists } y \in M \text{ such that } \overline{\mathcal{G}}(x') \neq \overline{\mathcal{G}}(y) \text{ and } x' + y \in T_z\}.$$

Applying (2.7) we deduce that whenever $\mu^2(E_\delta) < \delta$ there exist $b_z^\delta \in \mathbb{C}\backslash\{0\}$ and $v_z^\delta \in \mathbb{C}^d$ such that

$$f(x)(b_z^\delta \exp(v_z^\delta \cdot x))^{-1} = 1 + o_z(1) \tag{4.1}$$

for all $x \in \mathcal{N}_z$ except for a set of μ-measure $o_z(1)$.

The next lemma may be interpreted as an approximate analogue of Lemma 2.2.

Lemma 4.2. *Let M be a nowhere-flat bounded hypersurface with finite induced measure μ. Let $c \in \mathbb{C}\backslash\{0\}$ and $v \in \mathbb{C}^d$. Given $\epsilon > 0$ there exists $\delta = \delta(\epsilon, M) > 0$ such that if*

$$\mu(\{|c\exp(v \cdot x) - 1| > \delta\}) < \delta$$

then $|c-1|$, $\|v\|$ are less than ϵ.

Proof. Let $L_\delta = \{x \in M \mid |c\exp(v\cdot x)-1| \leq \delta\}$ and suppose that $\mu(M\backslash L_\delta) < \delta < 1$. For all $(x,y) \in L_\delta^2$ it follows that $|c^2\exp(v\cdot(x+y))-1| \leq 3\delta$. Since M is nowhere-flat, $\mu^2(\{(x,y) \in M^2 \mid \overline{\mathcal{G}}(x) = \overline{\mathcal{G}}(y)\}) = 0$ so

$$\mu^2(\{(x,y) \in M^2 \mid \overline{\mathcal{G}}(x) \neq \overline{\mathcal{G}}(y) \text{ and } x+y \in 2L_\delta\}) \geq \mu^2(L_\delta^2).$$

By Lemma 2.7, there exists a function $l : (0,1) \to (0,\infty)$ independent of c, v and δ such that $l(\delta) \to 0$ as $\delta \to 0$ and $\lambda^d(R_M\backslash R_M \cap 2L_\delta) < l(\delta)$. Thus for all $w \in R_M$ outside a set of λ^d-measure $l(\delta)$,

$$|c^2\exp(v\cdot w) - 1| \leq 3\delta.$$

Since R_M is open and non-empty, it follows that there exists a function $\epsilon : (0,1) \to (0,\infty)$ independent of c, v and δ such that $\epsilon(\delta) \to 0$ as $\delta \to 0$ and $|c-1|, \|v\| \leq \epsilon(\delta)$. ∎

Let $\epsilon > 0$ be fixed. Let $\tau > 0$ be a small positive parameter to be chosen below. Then there exists a set $\{z_1, \ldots, z_n\}$ of elements of M depending on τ such that $\mathcal{N} = \cup_{j=1}^n \mathcal{N}_{z_j}$ is connected and

$$\mu(M\backslash\mathcal{N}) < \tau.$$

By (4.1) and Lemma 4.2, for small $\delta > 0$ if $\mu^2(E_\delta) < \delta$ then for each $1 \leq i,j \leq n$, either the sets $\mathcal{N}_{z_i}$ and $\mathcal{N}_{z_j}$ are disjoint or

$$|b_{z_i}^\delta(b_{z_j}^\delta)^{-1} - 1|, \|v_{z_i}^\delta - v_{z_j}^\delta\| = o_\tau(1).$$

Set $b_\delta = b_{z_1}^\delta$ and $v_\delta = v_{z_1}^\delta$. Since $\mathcal{N}$ is connected, it follows that for each $1 \leq j \leq n$,

$$|b_\delta(b_{z_j}^\delta)^{-1} - 1|, \|v_\delta - v_{z_j}^\delta\| = o_\tau(1). \tag{4.2}$$

Observe that (4.2) implies that for all $x \in M$

$$(b_\delta\exp(v_\delta\cdot x))(b_{z_j}^\delta\exp(v_{z_j}^\delta\cdot x))^{-1} = 1 + o_\tau(1).$$

Therefore (4.1) implies that if $\mu(E_\delta) < \delta$ then there exist $b_\delta \in \mathbb{C}\backslash\{0\}$, $v_\delta \in \mathbb{C}^d$ and a set $\mathcal{N}^\delta \subset \mathcal{N}$ depending on τ such that $\mu(\mathcal{N}\backslash\mathcal{N}^\delta) = o_\tau(1)$ and

$$f(x)(b_\delta\exp(v_\delta\cdot x))^{-1} = 1 + o_\tau(1)$$

for all $x \in \mathcal{N}^\delta$. Choosing $\tau < \epsilon/2$, say, and taking δ sufficiently small completes the proof of Theorem 1.3. ∎

5. Multiplicative Functions on Curves

Let $I \subset \mathbb{R}$ be an open interval and suppose that $\gamma : I \to \mathbb{R}^d$ is a smooth isometric—that is, unit speed—embedding. Suppose that no open subset of $\Gamma = \gamma(I)$ lies in an affine hyperplane. This is equivalent to demanding that whenever $U \subset I$ is open, there exist $u_1, \ldots, u_d \in U$ such that the set of vectors $\{\dot\gamma(u_1), \ldots, \dot\gamma(u_d)\}$ is a basis for $\mathbb{R}^d$. Furthermore, arguing similarly to Lemma 2.2, this means that we may assume without loss of generality that I is bounded.

In this section we prove Theorem 1.5; the additive analogue follows similarly. For each $0 \leq j \leq d$, let $f_j : I \to \mathbb{C}$ be a measurable function which only vanishes on a λ-null set. Assume, in addition, that each f_j is locally λ-bounded in the sense that $\operatorname{ess\,sup}_{x\in K}|f_j(x)|$ is finite for each compact $K \subset I$. Suppose that there exists a measurable function $F : (d+1)\Gamma \to \mathbb{C}$ such that

$$\prod_{j=0}^{d} f_j(x_j) = F\big(\sum_{j=0}^{d} \gamma(x_j)\big) \text{ for } \lambda^{d+1}\text{-a.e. } (x_0, \ldots, x_d) \in I^{d+1}. \tag{5.1}$$

Define

$$\mathcal{A} = \{(x_0, \ldots, x_d) \in I^{d+1} \mid \operatorname{span}\{\dot\gamma(x_0)), \ldots, \dot\gamma(x_d)\} = \mathbb{R}^d\}.$$

The smooth map $I^{d+1} \to \mathbb{R}^d$ given by $(x_0, \ldots, x_d) \mapsto \sum_{j=0}^{d} \gamma(x_j)$ restricts to a submersion $\alpha : \mathcal{A} \to \mathbb{R}^d$. By Lemma 2.7 and (5.1), the assumed local boundedness of f_j for $0 \leq j \leq d$ implies that F is locally λ^d-integrable on the open set $\alpha(\mathcal{A}) \subset \mathbb{R}^d$. Choose $u_1, \ldots, u_d \in I$ such that $\{\dot\gamma(u_1), \ldots, \dot\gamma(u_d)\}$ is a basis for $\mathbb{R}^d$. By the inverse function theorem, the smooth map $\beta : I^d \to \mathbb{R}^d$ given by $(x_1, \ldots, x_d) \mapsto \sum_{j=1}^{d} \gamma(x_j)$ is a diffeomorphism from a bounded open neighbourhood $U \subset I^d$ of $(u_1, \ldots, u_d)$ onto a non-empty open ball $V = B_r(z) \subset \mathbb{R}^d$ and $I \times U \subset \mathcal{A}$. By shrinking U and V if necessary, we may assume that, in addition,

$$\chi = \int_U \big(\prod_{j=1}^{d} f_j(x_j)\big) dx_1 \ldots dx_d \neq 0.$$

Therefore f_0 agrees λ-almost everywhere with the continuous function

$$g_0 : x_0 \mapsto \frac{\int_U F\big(\gamma(x_0) + \sum_{j=1}^{d} \gamma(x_j)\big) dx_1 \ldots dx_d}{\int_U \big(\prod_{j=1}^{d} f_j(x_j)\big) dx_1 \ldots dx_d} \tag{5.2}$$

and similarly, for each $1 \leq j \leq d$, the function f_j agrees λ-almost everywhere with a continuous function g_j. By Lemma 2.7 and (5.1), $F|_{\alpha(\mathcal{A})}$ agrees almost everywhere with a continuous function G.

Thus,

$$g_0(x) = \chi^{-1} \int_V G(\gamma(x) + u) J_\beta(u) du = \chi^{-1} \int_{V+\gamma(x)} G(u) J_\beta(u - \gamma(x)) du$$

where J_β denotes the Jacobian of β. By the continuity of G and the smoothness of γ and J_β, it follows by the Leibniz integral theorem that g_0 is differentiable. Similarly,

for each $1 \le j \le d$, the function g_j is differentiable. Therefore G is differentiable and

$$\prod_{j=0}^{d} g_j(x_j) = G\big(\sum_{j=0}^{d} \gamma(x_j)\big) \text{ for all } (x_0, \ldots, x_d) \in \mathcal{A}. \tag{5.3}$$

Differentiating with respect to x_0,

$$g_0'(x_0) \prod_{j=1}^{d} g_j(x_j) = \dot\gamma(x_0) \cdot \nabla G\big(\sum_{j=0}^{d} \gamma(x_j)\big) \text{ for all } (x_0, \ldots, x_d) \in \mathcal{A}. \tag{5.4}$$

Fix $x_0 \in I$. Let $V = B_r(z)$ be the open ball defined above. Choose $v_1, \ldots, v_d \in \gamma^{-1}(B_r(\gamma(x_0)))$ such that $\dot\gamma(v_1), \ldots, \dot\gamma(v_d)$ span $\mathbb{R}^d$. Without loss of generality we may assume that $g_0(v_j) \neq 0$ for each $1 \le j \le d$. Let S be a non-empty connected open neighbourhood of $\gamma(x_0) + z$ contained in $\cap_{j=1}^d (\gamma(v_j) + B_r(z))$. By equation (5.4), for each $1 \le j \le d$,

$$\partial_{\dot\gamma(v_j)} G(s) = \frac{g_0'(v_j)}{g_0(v_j)} G(s) \text{ for all } s \in S \tag{5.5}$$

where $\partial_w G$ denotes the directional derivative of G in the direction w. Since the directions $\dot\gamma(v_1), \ldots, \dot\gamma(v_d)$ span $\mathbb{R}^d$, it follows that there exist $c_S \in \mathbb{C}$ and $\xi_S \in \mathbb{C}^d$ such that $G(s) = c_S \exp(\xi_S \cdot s)$ for all $s \in S$. Thus there exist a neighbourhood $B \subset I$ of x_0, $c_B \neq 0$ and $\xi_B \in \mathbb{C}^d$ such that for all $x \in B$, $g_0(x) = c_B \exp(\xi_B \cdot \gamma(x))$. Arguing as in the proof of Lemma 2.2 using the assumption that Γ is not contained in an affine hyperplane, this implies that there exist $c_0 \neq 0$ and $\xi_0 \in \mathbb{C}^d$ such that $g_0(x) = c_0 \exp(\xi_0 \cdot \gamma(x))$ for all $x \in I$. Moreover, c_0 and ξ_0 are uniquely determined.

This proves Theorem 1.5 in the case when each function f_j is locally bounded. Therefore to prove the theorem in its full generality it suffices to establish that the functional equation (5.1) forces measurable functions $f_0, \ldots, f_d$ to be locally bounded. This type of strategy is standard; it is discussed in [6], for example.

Suppose that the f_j and F are merely measurable and let U, V and the diffeomophism $\beta : U \to V$ be as defined above. By Lusin's theorem applied to the function $(x_1, \ldots, x_d) \mapsto \prod_{j=1}^d |f_j(x_j)|$, there exists a compact set $\mathcal{K} \subset U$ of positive λ^d-measure such that

$$\inf_{(x_1,\ldots,x_d)\in\mathcal{K}} \prod_{j=1}^{d} |f_j(x_j)| \ge c_0 > 0$$

for some positive constant c_0. There exists $I_1 \subset I$ such that $\lambda(I \backslash I_1) = 0$ and for each $x \in I_1$ the functional equation (5.1) is satisfied for the point $(x, x_1, \ldots, x_d)$ for λ^d-almost every $(x_1, \ldots, x_d) \in \mathcal{K}$.

By Lusin's theorem applied to $F|_{\gamma(I)+V}$, there exist a compact subset T of the open set $\gamma(I) + V$ and a constant $C < \infty$ such that

$$\lambda^d((\gamma(I) + V)\backslash T) < \lambda^d(\beta(\mathcal{K})) \neq 0, \tag{5.6}$$

and for all $w \in T$, $|F(w)| \le C$. Let $x \in I_1$. Then $\lambda^d(\gamma(x)+\beta(\mathcal{K})) = \lambda^d(\beta(\mathcal{K}))$ so (5.6) implies that $T \cap (\gamma(x) + \beta(\mathcal{K}))$ has positive λ^d-measure. Thus, there exists $w \in \mathcal{K}$ such that $|F(\gamma(x) + \beta(w))| \le C$. Hence $|f_0(x)| \le c_0^{-1} C$ and f_0 is λ-bounded. □

6. Three-fold Multiplicative Functions on Hypersurfaces

In this section we prove Theorem 1.7 by appropriately adapting the setup in §2 for suitably non-degenerate hypersurfaces and appealing to Theorem 1.5 with $d = 2$ for the remaining cases.

Let $d \geq 3$. Let $M \subset \mathbb{R}^d$ be a nowhere-flat hypersurface. By Lemma 2.2, we can assume without loss of generality that M is bounded and has finite induced measure μ; this lemma also implies the stated uniqueness.

Write points in $(\mathbb{R}^d)^6 \times (\mathbb{R}^d)^6 \times (\mathbb{R}^d)^6$ as (x, y, z) where $x = (x_1, \dots, x_6)$, $y = (y_1, \dots, y_6)$ and $z = (z_1, \dots, z_6)$ with $x_j, y_j, z_j \in \mathbb{R}^d$. Let Π be the hyperplane in $(\mathbb{R}^d)^6 \times (\mathbb{R}^d)^6 \times (\mathbb{R}^d)^6$ defined by the three equations

$$x_1 + y_2 + z_3 = x_4 + y_5 + z_6,$$
$$x_2 + y_3 + z_1 = x_5 + y_6 + z_4,$$
$$x_3 + y_1 + z_2 = x_6 + y_4 + z_5.$$

Let $\mathcal{P}_M = (M^6 \times M^6 \times M^6) \cap \Pi$. Write $\mathcal{S}_M$ for the set of smooth points of $\mathcal{P}_M$. Then $\mathcal{S}_M$ is a submanifold of $\mathbb{R}^{18d}$ of dimension $18(d-1) + 15d - 18d = 15d - 18$. Write σ for the volume measure on $\mathcal{S}_M$ associated to the induced Riemannian structure. Observe that the point $(x, y, z) \in \mathcal{P}_M$ lies in $\mathcal{S}_M$ if and only if

$$T_{x_1}M + T_{y_2}M + T_{z_3}M + T_{x_4}M + T_{y_5}M + T_{z_6}M = \mathbb{R}^d, \tag{6.1}$$
$$T_{x_2}M + T_{y_3}M + T_{z_1}M + T_{x_5}M + T_{y_6}M + T_{z_4}M = \mathbb{R}^d, \tag{6.2}$$
$$T_{x_3}M + T_{y_1}M + T_{z_2}M + T_{x_6}M + T_{y_4}M + T_{z_5}M = \mathbb{R}^d. \tag{6.3}$$

Consider the $5d$-dimensional hyperplane $\Lambda \subset (\mathbb{R}^d)^6$ defined by $w_1 + w_2 + w_3 = w_4 + w_5 + w_6$. The linear addition map $(\mathbb{R}^d)^6 \times (\mathbb{R}^d)^6 \times (\mathbb{R}^d)^6 \to (\mathbb{R}^d)^6$, $(x, y, z) \mapsto x + y + z$ restricts to a smooth map $\pi_M : \mathcal{S}_M \to \Lambda$. Write $\mathcal{R}_M$ for the set of regular points of π_M. Note that a necessary condition for $(x, y, z) \in \mathcal{S}_M$ to lie in $\mathcal{R}_M$ is that

$$T_{x_j}M + T_{y_j}M + T_{z_j}M = \mathbb{R}^d \text{ for all } 1 \leq j \leq 6. \tag{6.4}$$

For $S \subset M^3$, define $\mathcal{S}_M(S) = \{(x, y, z) \in \mathcal{S}_M \mid (x_i, y_j, z_k) \in S \text{ for all } 1 \leq i, j, k \leq 6\}$. Let $f_1, f_2, f_3 : M \to \mathbb{C}$, and $F : 3M \to \mathbb{C}$ be measurable functions. Suppose that

$$f_1(x)f_2(y)f_3(z) = F(x + y + z) \text{ for all } (x, y, z) \in S \subset M^3. \tag{6.5}$$

Then, similarly to the proof of Lemma 2.4, whenever $w \in \Lambda$ is in the image of $\mathcal{S}_M(S)$ under π_M,

$$F(z_1)F(z_2)F(z_3) = F(z_4)F(z_5)F(z_6).$$

The addition map $M^3 \to \mathbb{R}^d$ restricts to a smooth submersion $\alpha_M : \mathcal{M} \to R_M$ where we define $\mathcal{M} = \{(x, y, z) \in M^3 \mid T_xM + T_yM + T_zM = \mathbb{R}^d\}$, the set of regular points of the addition map, and $R_M = \{x + y + z \mid x, y, z \in M \text{ and } T_xM + T_yM + T_zM = \mathbb{R}^d\}$. Then R_M is an open subset of $\mathbb{R}^d$ which is dense in $3M$. By (6.4), we can also define the submersion $\gamma_M : \mathcal{R}_M \to R_M$ given by $(x, y, z) \mapsto x_1 + y_1 + z_1$. As before, a key ingredient will be the surjectivity of this submersion; we will assume that $\mathcal{G}(M) \subset \mathbb{S}^{d-1}$ is not contained in a planar circle so that M is not itself a cylinder. For the case when M is a cylinder, we will instead apply Theorem 1.5.

Proposition 6.1. *Let $M \subset \mathbb{R}^d$ be a nowhere-flat hypersurface such that the set $\mathcal{G}(M) \subset \mathbb{S}^{d-1}$ is not contained in a planar circle. Then $\gamma_M : \mathcal{R}_M \to R_M$ is surjective.*

Proof. Let $w \in R_M$. Write $w = x_1+y_1+z_1$ for $x_1, y_1, z_1 \in M$ such that $T_{x_1}M+T_{y_1}M+T_{z_1}M = \mathbb{R}^d$. By relabelling if necessary we may assume that $\overline{\mathcal{G}}(x_1) \neq \overline{\mathcal{G}}(y_1), \overline{\mathcal{G}}(z_1)$. Set $x_4 = x_1$, $y_4 = y_1$, $z_4 = z_1$.

Since $\mathcal{G}(M)$ is not contained in a planar circle, there exist $x_2, y_2 \in M$ such that $\mathcal{G}(x_2), \mathcal{G}(y_2) \notin \operatorname{span}\{\mathcal{G}(x_1), \mathcal{G}(z_1)\}$. Since M is nowhere-flat, we may take $\overline{\mathcal{G}}(x_2) \neq \overline{\mathcal{G}}(y_2)$. Define $\Delta = \operatorname{span}\{\mathcal{G}(x_1), \mathcal{G}(z_1)\} \cap \operatorname{span}\{\mathcal{G}(x_2), \mathcal{G}(y_2)\}$. Then $\dim \Delta \leq 1$. Set $z_2 = x_1$, $x_5 = x_2$, $y_5 = y_2$, $z_5 = z_2$.

Let $x_3 \in M$ be any element of M. Choose $y_3 \in M$ such that $\mathcal{G}(y_3) \notin \Delta$. Since $\mathcal{G}(M)$ is not contained in a planar circle, there exists $z_3 \in M$ such that $\mathcal{G}(z_3) \notin \operatorname{span}\{\mathcal{G}(y_3)\} + \Delta$. Then

$$\operatorname{span}\{\mathcal{G}(x_1), \mathcal{G}(z_1)\} \cap \operatorname{span}\{\mathcal{G}(x_2), \mathcal{G}(y_2)\} \cap \operatorname{span}\{\mathcal{G}(y_3), \mathcal{G}(z_3)\} = \{0\}. \tag{6.6}$$

Set $x_6 = x_3$, $y_6 = y_3$, $z_6 = z_3$.

Then

$$x_1 + y_2 + z_3 = x_4 + y_5 + z_6,$$

$$x_2 + y_3 + z_1 = x_5 + y_6 + z_4,$$

$$x_3 + y_1 + z_2 = x_6 + y_4 + z_5$$

so the point (x, y, z) for $x = (x_1, \ldots, x_6)$, $y = (y_1, \ldots, y_6)$, $z = (z_1, \ldots, z_6)$ lies in $\mathcal{P}_M$. By construction, (x, y, z) satisfies (6.1), (6.2) and (6.3) so $(x, y, z) \in \mathcal{S}_M$.

We will identify $T_{(x,y,z)}\mathcal{S}_M$ with $(\prod_{j=1}^6 T_{x_j}M \times \prod_{j=1}^6 T_{y_j}M \times \prod_{j=1}^6 T_{z_j}M) \cap \Pi$ and $T_{x+y+z}\Lambda$ with Λ in the canonical way. Then $(d\pi_M)_{(x,y,z)} : T_{(x,y,z)}\mathcal{S}_M \to T_{x+y+z}\Lambda$ is given by

$$(d\pi_M)_{(x,y,z)}(u, v, w) = u + v + w$$

where the tuples $u = (u_1, \ldots, u_6)$, $v = (v_1, \ldots, v_6)$, $w = (w_1, \ldots, w_6)$ satisfy $u_j \in T_{x_j}M$, $v_j \in T_{y_j}M$, $w_j \in T_{z_j}M$ for $1 \leq j \leq 6$ and the linear equations

$$u_1 + v_2 + w_3 = u_4 + v_5 + w_6,$$

$$u_2 + v_3 + w_1 = u_5 + v_6 + w_4,$$

$$u_3 + v_1 + w_2 = u_6 + v_4 + w_5.$$

Let $q \in \Lambda$, so $q_1 + q_2 + q_3 = q_4 + q_5 + q_6$. For each $1 \leq j \leq 6$, let (u_j, v_j, w_j) vary over the $(2d-3)$-dimensional affine subspace of $T_{x_j}M \times T_{y_j}M \times T_{z_j}M$ given by

$$u_j + v_j + w_j = q_j. \tag{6.7}$$

Set

$$p_1 = u_1 + w_1,\ p_2 = u_2 + v_2,\ p_3 = v_3 + w_3, \tag{6.8}$$

$$p_4 = u_4 + w_4,\ p_5 = u_5 + v_5,\ p_6 = v_6 + w_6. \tag{6.9}$$

Since $\overline{\mathcal{G}}(y_1) \neq \overline{\mathcal{G}}(x_1), \overline{\mathcal{G}}(z_1)$, condition (6.7) implies that p_1 may be varied freely in $(q_1 - T_{y_1}M) \cap (T_{x_1}M + T_{z_1}M) = q_1 + T_{y_1}M$ and then $v_1 = q_1 - p_1$ is determined. Similarly, p_2 may be varied freely over a certain translate of $T_{z_2}M$ and then $w_2 =$

$q_2 - p_2$ is determined, and so on for p_3, p_4, p_5, p_6 varying over translates of $T_{x_3}M$, $T_{y_4}M$, $T_{z_5}M$, $T_{x_6}M$ which then determine u_3, v_4, w_5, u_6 respectively. Since

$$T_{y_1}M + T_{z_2}M + T_{x_3}M - T_{y_4}M - T_{z_5}M - T_{x_6}M = \mathbb{R}^d,$$

as $p_1, \ldots, p_6$ vary, the vector $p_1 + p_2 + p_3 - p_4 - p_5 - p_6$ varies over all of $\mathbb{R}^d$. Fix an admissible set of $p_1, \ldots, p_6$ such that

$$p_1 + p_2 + p_3 = p_4 + p_5 + p_6. \tag{6.10}$$

Since $\overline{\mathcal{G}}(x_1) \neq \overline{\mathcal{G}}(z_1)$, the relation (6.8) implies that we may vary u_1 freely in a certain translate of the linear subspace $T_{x_1}M \cap T_{z_1}M$ and then $w_1 = p_1 - u_1$ is determined, and so on for v_2, w_3, u_4, v_5, w_6 varying over translates of $T_{y_2}M \cap T_{x_2}M$, $T_{z_3}M \cap T_{y_3}M$, $T_{x_4}M \cap T_{z_4}M$, $T_{y_5}M \cap T_{x_5}M$, $T_{z_6}M \cap T_{y_6}M$ which then determine u_2, v_3, w_4, u_5, v_6 respectively. By (6.6), it follows that

$$\begin{aligned} &T_{x_1}M \cap T_{z_1}M + T_{y_2}M \cap T_{x_2}M + T_{z_3}M \cap T_{y_3}M \\ &- T_{x_4}M \cap T_{z_4}M - T_{y_5}M \cap T_{x_5}M - T_{z_6}M \cap T_{y_6}M \\ &= T_{x_1}M \cap T_{z_1}M + T_{y_2}M \cap T_{x_2}M + T_{z_3}M \cap T_{y_3}M \\ &= \mathbb{R}^d. \end{aligned}$$

Therefore as u_1,v_2, w_3, u_4, v_5, w_6 vary, the vector $u_1 + v_2 + w_3 - u_4 - v_5 - w_6$ varies over all of $\mathbb{R}^d$; choose admissible u_1,v_2, w_3, u_4, v_5, w_6 such that

$$u_1 + v_2 + w_3 = u_4 + v_5 + w_6. \tag{6.11}$$

By (6.10) and (6.11),

$$\begin{aligned} u_2 + v_3 + w_1 &= (p_2 - v_2) + (p_3 - w_3) + (p_1 - u_1) \\ &= (p_4 + p_5 + p_6) - (u_4 + v_5 + w_6) = u_5 + v_6 + w_4 \end{aligned}$$

and then by (6.7),

$$\begin{aligned} u_3 + v_1 + w_2 &= (q_3 - p_3) + (q_1 - p_1) + (q_2 - p_2) \\ &= (q_4 + q_5 + q_6) - (p_4 + p_5 + p_6) = u_6 + v_4 + w_5. \end{aligned}$$

□

Suppose that the functions f_1, f_2, f_3, F above are measurable, that f_1, f_2, f_3 vanish only on a μ-null set and that $\mu^3(M^3 \backslash S) = 0$. Similarly to §2, by Proposition 6.1 it follows that for each $z \in R_M$, there exists a non-empty open ball $T_z \subset \mathbb{R}^d$ with center at z, translates B_z and C_z of T_z and a measurable function $H : T_z + B_z + C_z \to \mathbb{C}$ such that

$$H(w_1 + w_2 + w_3) = F(w_1)F(w_2)F(w_3)$$

for λ^{3d}-almost every triple $(w_1, w_2, w_3) \in T_z \times B_z \times C_z$. By fixing a typical w_3 and applying Lemma 2.1, it follows that there exist $c_z \in \mathbb{C}\backslash\{0\}$ and $v_z \in \mathbb{C}^d$ such that $F(w) = c_z \exp(v_z \cdot w)$ for λ^d-almost every $w \in T_z$.

Define an equivalence relation $\sim$ on R_M by declaring $w \sim u$ whenever $v_w = v_u$. This partitions R_M into a collection of open sets $\{R_j\}_{j \in J}$ such that each $z \in R_j$ has a neighbourhood on which F is of the form $F(z) = c \exp(v_j \cdot z)$, up to a null set, for some c depending on the neighbourhood.

For $j \in J$, consider the set M_j of all $x \in M$ for which there exist $y, z \in M$ such that $x + y + z \in R_j$. Since M is nowhere-flat, the sets $\{M_j\}_{j \in J}$ are an open cover for M. By Lemma 2.2, they form a partition for M. Assuming M is connected, it follows that $M_j = M$ for some $j \in J$. This means that each $x \in M$ has an open neighbourhood U in M on which f_1 is of the form $f_1(x) = c_U \exp(v_j \cdot x)$ except for a μ-null set for some constant $c_U \neq 0$. Since M is connected, it follows that there is a constant $c_1 \neq 0$ such that $f_1(x) = c_1 \exp(v_j \cdot x)$ for μ-almost every all $x \in M$. Thus the proof of Theorem 1.7 is complete for the case when $d \geq 3$ and $\mathcal{G}(M)$ is not contained in a planar circle.

Suppose now that $M \subset \mathbb{R}^d$ is any nowhere-flat hypersurface. The case $d = 2$ in Theorem 1.7 follows immediately from Theorem 1.5. For $d \geq 3$, each point of $x \in M$ has a small open neighbourhood U such that either U is a cylinder or $\mathcal{G}(U)$ is not contained in a planar circle. By Lemma 2.2, the following lemma therefore completes the proof of Theorem 1.7. The additive analogue of the theorem is, once again, similar.

Lemma 6.2. *Let $d \geq 3$. Suppose that $\Gamma \subset \mathbb{R}^2$ is a nowhere-flat bounded smooth embedded planar curve with finite induced measure γ and $V \subset \mathbb{R}^{d-2}$ is a non-empty open ball. Then the nowhere-flat hypersurface $M = \Gamma \times V \subset \mathbb{R}^d$ satisfies the following property. Suppose that $f_1, f_2, f_3 : M \to \mathbb{C}$ and $F : 3M \to \mathbb{C}$ are measurable such that f_1, f_2, f_3 vanish only on a $\gamma \times \lambda^{d-2}$-null set and for $(\gamma \times \lambda^{d-2})^3$-almost every $(x_1, x_2, x_3) \in M^3$,*
$$f_1(x_1)f_2(x_2)f_3(x_3) = F(x_1 + x_2 + x_3).$$
Then there exist $c \in \mathbb{C}\backslash\{0\}$ and $v \in \mathbb{C}^d$ such that
$$f_1(x) = c \exp(v \cdot x) \text{ for } \gamma \times \lambda^{d-2}\text{-a.e. } x \in M.$$

Proof. Write points in M as (t, y) for $t \in \Gamma$, $y \in V$. For $1 \leq j \leq 3$ and each $t \in \Gamma$, define $f_j^t : V \to \mathbb{C}$ by $f_j^t(y) = f_j(t, y)$ and for each $s \in 3\Gamma$ define $F^s : 3V \to \mathbb{C}$ by $F^s(z) = F(s, z)$. Then for γ^3-almost every $(t_1, t_2, t_3) \in \Gamma^3$, the functions $f_j^{t_j}$ for $1 \leq j \leq 3$ and $F^{t_1+t_2+t_3}$ are measurable, vanish only on a λ^{d-2}-null set and satisfy
$$f_1^{t_1}(y_1)f_2^{t_2}(y_2)f_3^{t_3}(y_3) = F^{t_1+t_2+t_3}(y_1 + y_2 + y_3)$$
for $\lambda^{3(d-2)}$-almost every $(y_1, y_2, y_3) \in V^3$.

By Lemma 2.1, it follows that for $1 \leq j \leq 3$ and γ^3-almost every $(t_1, t_2, t_3) \in \Gamma^3$,
$$f_j^{t_j}(y) = c_{j,t_j} \exp(v_{t_1+t_2+t_3} \cdot y) \text{ for } \lambda^{d-2}\text{-almost every } y \in V \tag{6.12}$$
and
$$F^{t_1+t_2+t_3}(z) = c_{t_1+t_2+t_3} \exp(v_{t_1+t_2+t_3} \cdot z) \text{ for } \lambda^{3(d-2)}\text{-almost every } z \in 3V,$$
where $c_{j,t_j}, c_{t_1+t_2+t_3} \in \mathbb{C}$ and $v_{t_1+t_2+t_3} \in \mathbb{C}^{d-2}$.

Thus for γ^3-almost every $(t_1, t_2, t_3) \in \Gamma^3$,
$$c_{1,t_1}c_{2,t_2}c_{3,t_3} = c_{t_1+t_2+t_3}.$$
Since f_j is measurable it follows that the function $t \mapsto c_{j,t}$ defined γ-almost everywhere is measurable for each $1 \leq j \leq 3$. Since F is measurable and Γ is nowhere-flat it follows that the functions $s \mapsto c_s$ and $s \mapsto v_s$ defined λ^2-almost everywhere on an

open subset of $\mathbb{R}^2$ which is dense in 3Γ are measurable. Since for each $1 \leq j \leq 3$ the left-hand side of (6.12) is independent of t_i, t_k for $\{i,k\} = \{1,2,3\}\backslash\{j\}$, it follows that for γ^3-almost every $(t_1,t_2,t_3) \in \Gamma^3$, the complex vector $v_{t_1+t_2+t_3} = v'$ is constant. By Theorem 1.5, it follows that there exist $b_j \in \mathbb{C}$ for $1 \leq j \leq 3$ and $v'' \in \mathbb{C}^2$ such that $c_{j,t} = b_j \exp(v'' \cdot t)$ for γ-almost every $t \in \Gamma$ for each $1 \leq j \leq 3$. Thus

$$f_j(x) = b_j \exp(v \cdot x) \text{ for } \gamma \times \lambda^{d-2}\text{-almost every } x \in M,$$

where $v = (v'', v') \in \mathbb{C}^d$. $\square$

References

[1] J. Aczel, *Lectures on Functional Equations and Their Applications*, Academic Press (1966).
[2] M. Christ and S. Shao, *On the extremizers of an adjoint Fourier restriction inequality*, Adv. in Math. 230 (2012), no.3, 957-977.
[3] M. Christ, *Near-extremizers of Young's inequality for* $\mathbb{R}^d$, pre-print, arXiv:1112.4875.
[4] D. Foschi, *Maximizers for the Strichartz Inequality*, J. Eur. Math. Soc. 9 (2007), no. 4.
[5] S. P. Ponomarev, *Submersions and preimages of sets of measure zero*, Siberian Math. Journal 28 (1987), 153-163.
[6] A. Jarai, *Regularity Properties of Functional Equations in Several Variables*, Adv. in Math., Vol. 8 (2005).

Marcos Charalambides, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

E-mail address: marcos@math.berkeley.edu